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Convergence of Martingales in Orlicz–Kantorovich Lattice

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ABSTRACT

In the present paper we prove convergence of martingales in Orlicz-Kantorovich lattices. Our main tool is the use of methods of measurable bundles of Banach-Kantorovich lattices.

Keywords: Orlicz-Kantorovich lattice, expectation operator, martingale

1. Introduction

The study of convergence of martingales in function spaces is one of important areas probability theory and functional analysis. Classical theory martingales is investigated by Doob (see (Doob, 1990)). The mean convergence of martingales in L_p–spaces developed by Helms in (Helms, 1958). Further development extends norm convergence of Banach space valued martingales in Orlicz spaces $L_M(\hat{\nabla}, X)$ is generalized by Uhl in (Uhl, 1969) as follows:

Let X be reflexive Banach space, N-function M meets the $\triangle_2$-condition and $\hat{\nabla}^{(n)}$ be an increasing sequence of regular Boolean subalgebras of $\hat{\nabla}$ and $\hat{\nabla}$ coincides with Boolean algebra $\bigcup\limits_{n=1}^{\infty} \hat{\nabla}^{(n)}$. The martingale $\{\hat{f}_n\}$ converges in $L_M(\hat{\nabla}, X)$ if and only if there exist $\hat{f} \in L_M(\hat{\nabla}, X)$ that $\hat{f}_n = E(\hat{f}|\hat{\nabla}^{(n)})$, where $E(\cdot|\hat{\nabla}^{(n)})$ is the conditional expectation operator on $L_M(\hat{\nabla}, X)$.

Norm convergence of martingales in rearrangement-invariant spaces are obtained in (Veksler, 1981), (Kikuchi, 2000a). Banach-Kantorovich lattices $L_p(\widehat{\nabla}, \widehat{\mu})$, constructed by measure $\widehat{\mu}$ with values in L_0- the space of measurable functions is found to have rich applications in analysis and were investigated in papers of Benderski (Benderskiy, 1976), Kusraev (**?**),(Kusraev, 2000) and Ganiev (Ganiev, 2006). One of the useful results, studied in (Ganiev, 2006), is that a Banach-Kantorovich lattice $L_p(\widehat{\nabla}, \widehat{\mu})$ can be represented as a measurable bundle of classical L_p – spaces. In (Zakirov and Chilin, 2009) this result has been extended for Orlicz —Kantorovich lattices $L_M(\widehat{\nabla}, \widehat{\mu})$. In (Ganiev, 2000) using a measurable bundle of classical L_p – spaces the convergence of martingales on Banach-Kantorovich $L_p(\widehat{\nabla}, \widehat{\mu})$ lattices is proved. In (Zakirov and Chilin, 2009) statistical and individual ergodic theorems has been proved for positive contractions of the Orlicz —Kantorovich lattices $L_M(\widehat{\nabla}, \widehat{\mu})$ in case when N-function M satisfies the condition

$$\sup_{s\geq 1} \left\{ \frac{1}{M(s)} \int\limits_1^s M(t^{-1}s)dt \right\} < \infty$$

In the present paper we prove convergence martingales in Orlicz-Kantorovich lattices.

Let $f = \{f_n\}_{n\geq 1}$ be martingale and $\{w_n\}_{n\geq 1}$ be a sequence of positive numbers such that $W_n = \sum\limits_{k=1}^{n} w_k \to \infty$. In (Kikuchi, 2000b) prove that a martingale $\hat{f} = \{\hat{f}_n\}_{n\geq 1}$ converges in rearrangement-invariant space Y (in partially in Orlicz spaces) if and only if $\frac{1}{W_n}\sum_{k=1}^{n} w_k f_k$ converges in Y. Kazamaki and Tsuchikura in (Kazamaki and Tsuchikura, 1967) proved this result for $L_p(1 < p < \infty)$– space.

We prove a similar theorem we prove for Orlicz–Kantorovich lattices $L_M(\widehat{\nabla}, \widehat{\mu})$.

Our main tool is the use of methods of measurable bundles of Orlicz-Kantorovich lattices.

2. Preliminaries

In this section we recall necessary definitions and results concerning Banach-Kantorovich lattices. Let (Ω, Σ, μ) be a space with complete finite measure, $L_0 = L_0(\Omega)$ be the algebra of classes of measurable functions on (Ω, Σ, μ).

Consider a real vector space E.

A transformation $\|\cdot\| : E \longrightarrow L_0$ is called vector-valued or L_0–valued norm on E, if it satisfies the following conditions:

1) $\|x\| \geq 0$ for all $x \in E$; $\|x\| = 0 \Longleftrightarrow x = 0$;

2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in R, x \in E$;

3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

A pair $(E, \|\cdot\|)$ is said to be a lattice-normed space(LNS) over L_0.

An LNS E is disjunctively decomposed or shortly, d — decomposed, if the following axiom is fulfilled : for any $x \in E$ and disjunct elements $e_1, e_2 \in L_0$, satisfying $\|x\| = e_1 + e_2$, there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$, $\|x_1\| = e_1$ and $\|x_2\| = e_2$.

A net $\{x_\alpha\} \in E$ is (bo)– convergent to $x \in E$, if a net $\{\|x_\alpha - x\|\}$ is (o) – convergent to L_0.

We say that an LNS is *(bo)*– complete, if any *(bo)* – fundamental net $\{x_\alpha\}$ *(bo)* – converges to some element of this space.

Any d — decomposable and *(bo)* – complete LNS over L_0 is said to be a

Banach-Kantorovich space (BKS) over L_0 (**?**).

If a Banach-Kantorovich space is simultaneously a vector lattice and the norm is monotone, then it becomes a *Banach-Kantorovich lattice.*

Let X be a mapping, which maps every point $\omega \in \Omega$ to some Banach space $(X(\omega), \|\cdot\|_{X(\omega)})$. In what follows, we assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function u is said to be a section of X, if it is defined almost everywhere in Ω and takes its value $u(\omega) \in X(\omega)$ for $\omega \in dom(u)$, where $\omega \in dom(u)$ is the domain of u.

Let L be some set of sections.

Definition 2.1. *(Gutman, 1995). A pair (X, L) is said to be a measurable bundle of Banach spaces over Ω if*

1. $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in dom(c_1) \cap dom(c_2) \to \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;

2. the function $\|c\| : \omega \in dom(c) \to \|c(\omega)\|_{X(\omega)}$ is measurable for all $c \in L$;

3. for every $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in dom(c)\}$ is dense in $X(\omega)$;

A measurable Banach bundle (X, L) is called measurable bundle of Banach lattices(MBBL), if $(X(\omega), \|\cdot\|_{X(\omega)})$ are Banach lattices for all $\omega \in \Omega$ and all $c_1, c_2 \in L$ $c_1 \vee c_2 \in L$, where $c_1 \vee c_2$: $\omega \in$ dom $(c_1) \cap$ dom $(c_2) \to c_1(\omega) \vee c_2(\omega)$.

Henceforth, (X, L) will be denoted just by X.

A section s is a step-section, if there are pairwise disjoint sets $A_1, A_2, \ldots, A_n \in \Sigma$ and sections $c_1, c_2, \ldots, c_n \in L$ such that $\bigcup_{i=1}^{n} A_i = \Omega$ and

$$s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) c_i(\omega)$$

for almost all $\omega \in \Omega$.

A section u is measurable, if for any $A \in \Sigma$ there is a sequence s_n of step-sections such that $s_n(\omega) \to u(\omega)$ for almost all $\omega \in A$.

Let $M(\Omega, X)$ be the set of all measurable sections. By symbol $L_0(\Omega, X)$ we denote factorization of the $M(\Omega, X)$ with respect to almost everywhere equality.

Usually, by $\hat{u}$ we denote a class from $L_0(\Omega, X)$, containing the section $u \in M(\Omega, X)$, and by $\|\hat{u}\|$ we denote the element of $L_0(\Omega)$, containing $\|u(\omega)\|_{X(\omega)}$.

Let X be an MBBL. We set $\hat{u} \leq \hat{v}$, if $u(\omega) \leq v(\omega)$ a.e. One can easily show that the relation $\hat{u} \leq \hat{v}$ constitutes a partial order on $L_0(\Omega, X)$.

If X is an MBBL, then $L_0(\Omega, X)$ is a Banach-Kantorovich lattice (Ganiev, 2006).

Let ∇_ω, $\omega \in \Omega$ be a family of complete boolean algebras with strictly positive real-valued measures μ_ω. We set $\rho_\omega(e, g) = \mu_\omega(e \bigtriangleup g)$, $e, g \in \nabla_\omega$. Then $(\nabla_\omega, \mu_\omega)$are complete metric spaces. Consider the transformation ∇, which assigns some boolean algebra ∇_ω to every point $\omega \in \Omega$. Let L be a non-empty set of sections ∇.

Definition 2.2. *A pair (∇, L) is called a measurable bundle of boolean algebras over Ω, if*

1) (∇, L) is a measurable bundle of metric spaces (Ganiev, 2006);
2) if $e \in L$, then $e^\perp \in L$, where $e^\perp : \omega \in \text{dom}\ (e) \to e^\perp(\omega)$;
3) if $e_1, e_2 \in L$, then $e_1 \vee e_2 \in L$, where $e_1 \vee e_2 : \omega \in \text{dom}\ (e_1) \cap \text{dom}\ (e_2) \to e_1(\omega) \vee e_2(\omega)$.

Let $M(\Omega, \nabla)$ be the set of measurable sections, $\hat{\nabla}-$ factorization of $M(\Omega, \nabla)$ with respect to almost everywhere equality.

Define a transformation $\hat{\mu} : \hat{\nabla} \to L_0(\Omega)$ by $\hat{\mu}(\hat{e}) = \hat{f}$, where $\hat{f}$ is a class containing the function $f(\omega) = \mu_\omega(e(\omega))$. Evidently, $\hat{\mu}$ is well defined.

It is well known that $(\hat{\nabla}, \hat{\mu})$ is a complete boolean algebra with strictly positive $L_0(\Omega)$ – valued modulated measure $\hat{\mu}$, moreover, the boolean algebra $\nabla(\Omega)$ of all idempotents from $L_0(\Omega)$ is identified with regular sub-algebra in $\hat{\nabla}$ and $\widehat{\mu}(g\widehat{e}) = g\widehat{\mu}(\widehat{e})$ for all $g \in \nabla(\Omega)$ and $\widehat{e} \in \widehat{\nabla}$.

By $L_0(\hat{\nabla}, \hat{\mu})$ we denote an order complete vector lattice $C_\infty(Q(\hat{\nabla}))$, where $Q(\hat{\nabla})$ is the Stonian compact associated with complete Boolean algebra $\hat{\nabla}$. Following the well known scheme of the construction of L_p-spaces, a space $L_p(\hat{\nabla}, \hat{\mu})$ can be defined by

$$L_p(\hat{\nabla}, \hat{\mu}) = \left\{ \hat{f} \in L_0(\hat{\nabla}, \hat{\mu}) : \int |\hat{f}|^p d\hat{\mu} - \text{exist} \right\}, \quad p \geq 1$$

where $\hat{\mu}$ is an $L_0(\Omega)$-valued measure on $\hat{\nabla}$.

It is known (**?**) that $L_p(\hat{\nabla},\hat{\mu})$ is a BKS over $L_0(\Omega)$ with respect to the $L_0(\Omega)$-valued norm $\|\hat{f}\|_{L_p(\hat{\nabla},\hat{\mu})} = \left(\int |\hat{f}|^p d\hat{\mu}\right)^{1/p}$. Moreover, $L_p(\hat{\nabla},\hat{\mu})$ is a Banach-Kantorovich lattice (see (Kusraev, 2000),(Ganiev, 2006)).

An even continuous convex function $M : R \to [0,\infty)$ is called an N-function, if $\lim\limits_{t\to 0} \frac{M(t)}{t} = 0$ and $\lim\limits_{t\to\infty} \frac{M(t)}{t} = \infty$. An N-function M is said to satisfy $\triangle_2$-condition on $[s_0,\infty), s_0 \geq 0$, if there exists a constant k such that $M(2s) \leq kM(s)$ for every $s \geq s_0$ (see (Krasnoselskii and Rutitski, 1961)).

The set

$$L_M^0 := L_M^0(\hat{\nabla},\hat{\mu}) := \{x \in L_0(\hat{\nabla}) : M(x) \in L_1(\hat{\nabla},\hat{\mu})\}$$

is called the Orlicz L_0-class, and the vector space

$$L_M := L_M(\hat{\nabla},\hat{\mu}) := \{x \in L_0(\hat{\nabla},\hat{\mu}) : xy \in L_1(\hat{\nabla},\hat{\mu}) for \quad all \quad y \in L_N^0\}$$

is called the Orlicz L_0-space, where N is the complementary N-function to M.

We notice that $L_M(\hat{\nabla},\hat{\mu}) \subset L_1(\hat{\nabla},\hat{\mu})$.

Define the L_0-valued Orlicz norm on $L_M(\hat{\nabla},\hat{\mu})$ as follows

$$\|x\|_M := \sup\left\{\left|\int xyd\hat{\mu}\right| : y \in A(N)\right\}, x \in L_M(\hat{\nabla},\hat{\mu}),$$

where $A(N) = \{y \in L_N^0 : \int N(y)d\hat{\mu} \leq \mathbf{1}\}$ and $\mathbf{1}$ is identity element of L_0. The pair $(L_M(\hat{\nabla},\hat{\mu}),\|\cdot\|_M)$ is a Banach–Kantorovich lattice which is called the Orlicz–Kantorovich lattice associated with the L_0-valued measure (Zakirov and Chilin, 2009).

Theorem 2.1. *(Zakirov and Chilin, 2009). If the N-function M meets the $\triangle_2$-condition then the Orlicz —Kantorovich lattice $L_M(\hat{\nabla},\hat{\mu})$ is isometrically and order isomorphic to $L_0(\Omega, X)$, where (X, L) is the measurable Banach bundle over Ω such that $X(\omega) = L_M(\nabla_\omega,\mu_\omega)$ and*

$$L = \left\{\sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{R}, \quad e_i \in M(\Omega,\nabla), \ i = \overline{1,n}, \ n \in \mathbb{N}\right\}$$

In Theorem 4.2.9 (**?**) it is proven that there exists conditionally expectation operator $E(\cdot|\hat{\nabla}^1) : L_1(\hat{\nabla},\hat{\mu}) \to L_1(\hat{\nabla}^1,\hat{\mu}^1)$, with respect regular subalgebra $\hat{\nabla}^1$ of $\hat{\nabla}$.

In this case $\|E(\hat{f}|\hat{\nabla}^1)\|_{L_1(\hat{\nabla},\hat{\mu})} \leq \|\hat{f}\|_{L_1(\hat{\nabla},\hat{\mu})}$ for any $\hat{f} \in L_1(\hat{\nabla},\hat{\mu})$ and $E(\mathbf{1}|\hat{\nabla}^1) = \mathbf{1}$.

Let Banach-Kantorovich lattice $L_p(\widehat{\nabla},\widehat{\mu})$ be represented as a measurable bundle of classical $L_p(\nabla_\omega,\mu_\omega)$–lattices. The description of conditionally expectation operator $E(\cdot|\hat{\nabla}^1) : L_1(\hat{\nabla},\hat{\mu}) \to L_1(\hat{\nabla}^1,\hat{\mu}^1)$ is obtained in (Ganiev, 2006):

Theorem 2.2. *Let $E(\cdot|\hat{\nabla}^1) : L_1(\hat{\nabla},\hat{\mu}) \to L_1(\hat{\nabla}^1,\hat{\mu}^1)$ be conditionally expectation operator. Then for any $\omega \in \Omega$ there exists $E_\omega(\cdot|\nabla^1_\omega) : L_1(\nabla_\omega,\mu_\omega) \to L_1(\nabla^1_\omega,\mu^1_\omega)$ conditionally expectation operator, such that*

$$E(\hat{f}|\hat{\nabla}^1)(\omega) = E_\omega(f(\omega)|\nabla^1_\omega)$$

for any $\hat{f} \in L_1(\hat{\nabla},\hat{\mu}$ and μ-a.e. $\omega \in \Omega$, where $E_\omega(\cdot|\nabla^1_\omega)$ is conditionally expectation operator on $L_p(\nabla_\omega,\mu_\omega)$.

3. Convergence martingales in Orlicz-Kantorovich lattices

In this section we are going to prove the main result of the paper. We will prove the convergence of martingales in Orlicz-Kantorovich lattice.

Proposition 3.1. *Let M be an N-function, and $E(\cdot|\hat{\nabla}^1) : L_1(\hat{\nabla},\hat{\mu}) \to L_1(\hat{\nabla}^1,\hat{\mu}^1)$ be conditionally expectation operator. Then*

$$E(L_M(\hat{\nabla},\hat{\mu})|\hat{\nabla}^1) \subset L_M(\hat{\nabla},\hat{\mu})$$

and $\|E(\cdot|\hat{\nabla}^1)\|_{L_M(\hat{\nabla},\hat{\mu})\to L_M(\hat{\nabla},\hat{\mu})} = \mathbf{1}$.

Proof. Since $\|E(\hat{f}|\hat{\nabla}^1)\|_{L_1(\hat{\nabla},\hat{\mu})} \leq \|\hat{f}\|_{L_1(\hat{\nabla},\hat{\mu})}$ for any $\hat{f} \in L_1(\hat{\nabla},\hat{\mu})$ and $E(\mathbf{1}|\hat{\nabla}^1) = \mathbf{1}$ by Proposition 3.1 (Zakirov and Chilin, 2009) $E(L_M(\hat{\nabla},\hat{\mu})|\hat{\nabla}^1) \subset L_M(\hat{\nabla},\hat{\mu})$.

As $\|E(\hat{f}|\hat{\nabla}^1)\|_M(\omega) = \|E(\hat{f}|\hat{\nabla}^1)(\omega)\|_{L_M(\nabla_\omega,\mu_\omega)} = \|E_\omega(f(\omega)|\nabla^1_\omega)\|_{L_M(\nabla_\omega,\mu_\omega)} \leq \|f(\omega)\|_{L_M(\nabla_\omega,\mu_\omega)} = \|\hat{f}\|_M(\omega)$ a.e. we get

$$\|E(\hat{f}|\hat{\nabla}^1)\|_M \leq \|\hat{f}\|_M$$

or

$$\|E(\cdot|\hat{\nabla}^1)\|_{L_M(\hat{\nabla},\hat{\mu})\to L_M(\hat{\nabla},\hat{\mu})} \leq \mathbf{1}.$$

As $\|E_\omega(f(\omega)|\nabla^1_\omega)\|_{L_M(\nabla_\omega,m_\omega)} = \|f(\omega)\|_{L_M(\nabla_\omega,\mu_\omega)}$ for almost all $\omega \in \Omega$ and for any $\{f(\omega)\}_{\omega\in\Omega} = \widehat{f}$ with $f(\omega) \in L_M(\nabla^1_\omega, \mu^1_\omega)$ we have that

$$\|E(\cdot|\hat{\nabla}^1)\|_{L_M(\hat{\nabla},\hat{\mu})\to L_M(\hat{\nabla},\hat{\mu})} = \mathbf{1}.$$

□

Let $\hat{\nabla}^{(n)}$ be an increasing sequence of regular Boolean sub-algebras from $\hat{\nabla}$, and $\hat{f}_n$ is a sequence belonging to $L_M(\hat{\nabla}, \hat{\mu})$, such that $\hat{f}_n \in L_M(\hat{\nabla}^{(n)}, \hat{\mu}^{(n)})$.

Definition 3.1. *The sequence $\{\hat{f}_n\}$ is called a martingale in the Orlicz-Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$ with respect to $\{\hat{\nabla}^{(n)}\}$, if for $n < m$ the equality*

$$E(\hat{f}_m|\hat{\nabla}^{(n)}) = \hat{f}_n$$

holds.

If $\hat{\nabla}^{(1)} \subset \hat{\nabla}^{(2)} \subset \hat{\nabla}$, then $E(E(\hat{f}|\hat{\nabla}^{(2)})|\hat{\nabla}^{(1)}) = E(\hat{f}|\hat{\nabla}^{(1)})$, therefore, $E(\hat{f}|\hat{\nabla}^{(n)})$ is an example of martingale in the Orlicz-Kantorivch lattice $L_M(\hat{\nabla}, \hat{\mu})$.

Theorem 3.1. *$\hat{\nabla}^{(n)}$ be an increasing sequence of regular Boolean subalgebras of $\hat{\nabla}$ and $\hat{\nabla}$ coincides with Boolean algebra $\bigcup\limits_{n=1}^{\infty} \hat{\nabla}^{(n)}$. The martingale $\{\hat{f}_n\}$ converges in $L_p(\hat{\nabla}, \hat{\mu})$ if and only if there exists $\hat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ that $\hat{f}_n = E(\hat{f}|\hat{\nabla}^{(n)})$.*

Proof. Let $\{\hat{f}_n\}$ be martingale and converges to $\widetilde{f}$ in $L_p(\hat{\nabla}, \hat{\mu})$. Then $\|f_n(\omega) - f(\omega)\|_{L_p(\nabla_\omega,\mu_\omega)} \to 0$ for almost all $\omega \in \Omega$. By Theorem II.4.2. (Vakhania et al., 1987) there exist $f(\omega) \in L_p(\nabla_\omega, \mu_\omega)$ such that

$$f_n(\omega) = E_\omega(f(\omega)|\nabla^{(n)}_\omega).$$

Since $f_n(\omega) = E_\omega(f(\omega)|\nabla^{(n)}_\omega) \to f(\omega)$ for almost all $\omega \in \Omega$ in $L_p(\nabla_\omega, \mu_\omega)$ (see (Vakhania et al., 1987). Theorem II.4.1.) we get f is measurable section and $f(\omega) = \widetilde{f}(\omega)$ a.e. Hence $f_n(\omega) = E_\omega(\widetilde{f}(\omega)|\nabla^{(n)}_\omega)$ and $\hat{f}_n = E_\omega(\widehat{\widetilde{f}(\omega)|\nabla^{(n)}_\omega}) = E(\widetilde{f}|\hat{\nabla}^{(n)})$, i.e. $\widehat{f} = \widetilde{f}$. The converse part is proven in (Ganiev, 2000). □

Theorem 3.2. *Let $L_M(\hat{\nabla}, \hat{\mu})$ be the Orlicz-Kantorovich lattice and the $N-$function M meets $\triangle_2-$condition, $\hat{\nabla}^{(n)}$ be an increasing sequence of regular Boolean subalgebras of $\hat{\nabla}$ and $\hat{\nabla}$ coincides with Boolean algebra $\bigcup\limits_{n=1}^{\infty} \hat{\nabla}^{(n)}$. Then the martingale $E(\hat{f}|\hat{\nabla}^{(n)})$ is (bo)-convergent in $L_M(\hat{\nabla}, \hat{\mu})$ for any $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$.*

Proof. As the N–function M meets $\triangle_2$–condition by Corollary 3.9 (Uhl, 1969) (see also Corollary 4.2 (Kikuchi, 2000a)) $E_\omega(f(\omega)|\nabla_\omega^n)$ is convergent in $L_M(\nabla_\omega, \mu_\omega)$. Since $\|\hat{f}\|_M(\omega) = \|f(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)}$ for almost all $\omega \in \Omega$ we get $E(\hat{f}|\hat{\nabla}^{(n)})$ is (bo)-convergent in $L_M(\hat{\nabla}, \hat{\mu})$. $\square$

Theorem 3.3. *Let $L_M(\hat{\nabla}, \hat{\mu})$ be the Orlicz-Kantorovich lattice and the N–function M meets $\triangle_2$–condition, $\hat{\nabla}^{(n)}$ be an increasing sequence of regular Boolean subalgebras of $\hat{\nabla}$ and $\hat{\nabla}$ coincides with Boolean algebra $\bigcup_{n=1}^{\infty} \hat{\nabla}^{(n)}$. If a martingale $\{\hat{f}_n\}$ is (bo)-convergent in $L_M(\hat{\nabla}, \hat{\mu})$ then there exist $\widetilde{f} \in L_M(\hat{\nabla}, \hat{\mu})$ that $\hat{f}_n = E(\widetilde{f}|\hat{\nabla}^{(n)})$.*

Proof. Let $\{\hat{f}_n\}$ be a martingale and convergent to $\widetilde{f}$ in $L_M(\hat{\nabla}, \hat{\mu})$. As the N–function M meets $\triangle_2$–condition by Theorem 3.8 (Uhl, 1969), then

$$f_n(\omega) = E_\omega(\widetilde{f}(\omega)|\nabla_\omega^{(n)}).$$

Hence $\hat{f}_n = E_\omega(\widehat{\widetilde{f}(\omega)|\nabla_\omega^{(n)}}) = E(\widetilde{f}|\hat{\nabla}^{(n)})$. $\square$

Theorem 3.4. *Let $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ and the N–function M has property*

$$\sup_{s \geq 1} \{ \frac{1}{M(s)} \int_1^s M(t^{-1}s)dt \} < \infty,$$

$\hat{\nabla}^{(n)}$ be an increasing sequence of regular Boolean subalgebras of $\hat{\nabla}$ and $\hat{\nabla}$ coincides with Boolean algebra $\bigcup_{n=1}^{\infty} \hat{\nabla}^{(n)}$. Then the martingale $E(\hat{f}|\hat{\nabla}^{(n)})$ is order bounded in the Orlicz-Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$ and $E(\hat{f}|\hat{\nabla}^{(n)})$ is (o)-convergent in $L_M(\hat{\nabla}, \hat{\mu})$.

Proof. Let $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ and $E(\hat{f}|\hat{\nabla}^n)(\omega) = E_\omega(f(\omega)|\nabla_\omega^n)$ a.e. From Corollary 1.3 (Braverman et al., 1998) follows that $L_M(\nabla_\omega, \mu_\omega)$ has the Hardy-Littlewood property (see (Braverman and Mekler, 1977, Braverman et al., 1998)). Hence from Theorem 3.1 (Braverman et al., 1998) follows that

$$\sup_{n \geq 1} E_\omega(f(\omega)|{\nabla_\omega}^{(n)})$$

exists in $L_M(\nabla_\omega, \mu_\omega)$. Then according Theorem 4.1 (Ganiev, 2006) and Proposition 2.3 (Zakirov and Chilin, 2009)

$$\sup_{n \geq 1} E(\widehat{f}|\widehat{\nabla}^{(n)}) \in L_M(\hat{\nabla}, \hat{\mu}).$$

Since $\hat{f} \in L_M(\hat{\nabla}, \hat{\mu}) \subset L_1(\hat{\nabla}, \hat{\mu})$ by (Ganiev, 2000) we get $E(\hat{f}|\hat{\nabla}^{(n)})$ is (o)-convergent in $L_0(\hat{\nabla})$. As $\sup\limits_{n\geq 1} E(\widehat{f}|\widehat{\nabla}^{(n)}) \in L_M(\hat{\nabla}, \hat{\mu})$, $E(\hat{f}|\hat{\nabla}^{(n)})$ is (o)-convergent in $L_M(\hat{\nabla}, \hat{\mu})$. □

Theorem 3.5. *Let $L_M(\hat{\nabla}, \hat{\mu})$ be the Orlicz-Kantorovich lattice and the N–function M meets $\triangle_2$–condition. If $\{\hat{f}_n\}$ is martingale bounded in $L_M(\hat{\nabla}, \hat{\mu})$, then $\{\hat{f}_n\}$ (bo)–converges in $L_M(\hat{\nabla}, \hat{\mu})$.*

Proof. Since $\sup\limits_{n\geq 1} \|\hat{f}_n\|_M$ exists, then $\sup\limits_{n\geq 1} \|f_n(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)} < \infty$ for almost all $\omega \in \Omega$. Then by Corollary 4.2 (Kikuchi, 2000a) implies $\|f_n(\omega) - f(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)} \to 0$ for almost all $\omega \in \Omega$ and for some $f(\omega) \in L_M(\nabla_\omega, \mu_\omega)$, i.e.

$$\|f_n\widehat{(\omega) - f}(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)} \xrightarrow{(o)} 0$$

Let $\hat{f} = \widehat{f(\omega)}$. As $\|\hat{f}_n - \hat{f}\|_M = \|f_n\widehat{(\omega) - f}(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)}$ then $\{\hat{f}_n\}$ (bo)–converges in $L_M(\hat{\nabla}, \hat{\mu})$. □

4. Convergence of weighted average of martingales in Orlicz-Kantorovich lattices

In this section we will prove the convergence of weighted average of martingales in Orlicz-Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$.

Definition 4.1. *(Kikuchi, 2000b). The sequence $\{w_n\}_{n\geq 1}$ of positive numbers is called a weight sequence if $W_n = \sum\limits_{k=1}^{n} w_k \to \infty$ as $n \to \infty$. The weighted average $\sigma_n(\widehat{f})$ of a martingale $\hat{f} = \{\hat{f}_n\}_{n\geq 1}$ given by $\sigma_n(\widehat{f}) = \frac{1}{W_n} \sum\limits_{k=1}^{n} w_k f_k$.*

Theorem 4.1. *Let $L_M(\hat{\nabla}, \hat{\mu})$ be the Orlicz-Kantorovich lattice and the N–function M meets $\triangle_2$–condition, $\widehat{f} = \{\widehat{f}_n\}_{n\geq 1}$ be a martingale in $L_M(\hat{\nabla}, \hat{\mu})$. Then $\{\widehat{f}_n\}_{n\geq 1}$ (bo)–converges in $L_M(\hat{\nabla}, \hat{\mu})$ if and only if $\sigma_n(\widehat{f})$ (bo)–converges in Orlicz-Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$.*

Proof. Let $\hat{f}_n \to \widetilde{f}$ in $L_M(\hat{\nabla}, \hat{\mu})$. Then $\|f_n(\omega) - \widetilde{f}(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)} \to 0$ for almost all $\omega \in \Omega$. Since

$$\|E_\omega(g(\omega)|\nabla_\omega^{(n)})\|_{L_M(\nabla_\omega, \mu_\omega)} \leq \|g(\omega)\|_{L_M(\nabla_\omega, \mu_\omega)}$$

we get

$$\sup_{n\geq 1} \|E_\omega(\cdot|\nabla_\omega^{(n)})\|_{L_M(\nabla_\omega,\mu_\omega)\to L_M(\nabla_\omega,\mu_\omega)} \leq 1$$

for almost all $\omega \in \Omega$. Then by Theorem 2 (Kikuchi, 2000b)

$$\sigma_n(f(\omega)) \to f'(\omega)$$

in $L_M(\nabla_\omega, \mu_\omega)$ for almost all $\omega \in \Omega$ and it is clear that f' is measurable section.

Since $\|\widehat{g}\|_M(\omega) = \|g(\omega)\|_{L_M(\nabla_\omega,\mu_\omega)}$ for almost all $\omega \in \Omega$ and for any $\widehat{g} \in L_M(\hat{\nabla}, \hat{\mu})$

$$\|\sigma_n(\widehat{f}) - \widehat{f}'\|_M(\omega) = \|\sigma_n(f(\omega) - f'(\omega)\|_{L_M(\nabla_\omega,\mu_\omega)} \to 0$$

for almost all $\omega \in \Omega$. Hence

$$\|\sigma_n(\widehat{f}) - \widehat{f}'\|_M \stackrel{(o)}{\to} 0,$$

i.e. $\sigma_n(\widehat{f})$ (bo)–converges in Orlicz–Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$.

Let $\sigma_n(\widehat{f})$ (bo)–converges in Orlicz–Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$. Then $\sigma_n(f(\omega))$ converges in $L_M(\nabla_\omega, \mu_\omega)$ for almost all $\omega \in \Omega$. Since

$$\sup_{n\geq 1} \|E_\omega(\cdot|\nabla_\omega^{(n)})\|_{L_M(\nabla_\omega,\mu_\omega)\to L_M(\nabla_\omega,\mu_\omega)} \leq 1$$

by Theorem 2 (Kikuchi, 2000b) $f_n(\omega)$ converges in $L_M(\nabla_\omega, \mu_\omega)$ for almost all $\omega \in \Omega$. Therefore, $\hat{f}_n = \widehat{f_n(\omega)}$ (bo)–converges in Orlicz–Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$.

□

Corollary 4.1. *Let $\widehat{f} = \{\widehat{f}_n\}_{n\geq 1}$ be a martingale in $L_p(\hat{\nabla}, \hat{\mu}), 1 \leq p < \infty$. Then $\{\widehat{f}_n\}_{n\geq 1}$ (bo)–converges in $L_p(\hat{\nabla}, \hat{\mu})$ if and only if $\sigma_n(\widehat{f})$ (bo)–converges in Banach–Kantorovich lattice $L_p(\hat{\nabla}, \hat{\mu})$.*

Now we generalize Theorem 4.1 and Corollary 4.1 to the summability of martingales by Toeplitz matrix. Let $(a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix of real numbers and (a_{ij}) a regular, i.e. (i) $\sum\limits_{j=1}^{\infty} |a_{ij}| < \infty$ for every i; (ii) $\lim\limits_{i\to\infty} a_{ij} = 0$ for every j; (iii) $\lim\limits_{i\to\infty} \sum\limits_{j=1}^{\infty} a_{ij} = 1$.

For a martingale $\widehat{f} = \{\widehat{f}_n\}_{n\geq 1}$, we put

$$T_i\widehat{f} = \sum_{j=1}^{\infty} a_{ij}\widehat{f}_j.$$

Theorem 4.2. *Let $L_M(\hat{\nabla}, \hat{\mu})$ be the Orlicz-Kantorovich lattice and the $N-$function M meets $\triangle_2-$condition, $\widehat{f} = \{\widehat{f_n}\}_{n\geq 1}$ be a martingale in $L_M(\hat{\nabla}, \hat{\mu})$. Then $\{\widehat{f_n}\}_{n\geq 1}$ $(bo)-$converges in $L_M(\hat{\nabla}, \hat{\mu})$ if and only if $T_i\widehat{f} \in L_M(\hat{\nabla}, \hat{\mu})$ and $T_i\widehat{f}$ $(bo)-$converges in Orlicz–Kantorovich lattice $L_M(\hat{\nabla}, \hat{\mu})$.*

The proof follows stalkwise using Theorem 4 (Kikuchi, 2000a).

Corollary 4.2. *Let $L_p(\hat{\nabla}, \hat{\mu}), 1 \leq p < \infty$ be the Banach–Kantorovich lattice and $\widehat{f} = \{\widehat{f_n}\}_{n\geq 1}$ be a martingale in $L_p(\hat{\nabla}, \hat{\mu})$. Then $\{\widehat{f_n}\}_{n\geq 1}$ $(bo)-$converges in $L_p(\hat{\nabla}, \hat{\mu})$ if and only if $T_i\widehat{f} \in L_p(\hat{\nabla}, \hat{\mu})$ and $T_i\widehat{f}$ $(bo)-$converges in Banach–Kantorovich lattice $L_p(\hat{\nabla}, \hat{\mu})$.*

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